Miscellaneous Calculus Problems

1. **(a)** Let $y = \pi - x$, $I = \int_0^\pi x f(\sin x)dx = \int_\pi^0 (\pi - y) f(\sin(\pi - y))(-dy) = \int_0^\pi (\pi - y) f(\sin y)dy$

$$= \pi\int_0^\pi f(\sin y)dy - \int_0^\pi y f(\sin y)dy = \pi\int_0^\pi f(\sin x)dx - \int_0^\pi x f(\sin x)dx = \pi\int_0^\pi f(\sin x)dx - I$$

$\therefore$ $2I = \pi\int_0^\pi f(\sin x)dx$ and $I = \frac{\pi}{2}\int_0^\pi f(\sin x)dx$

(b) $\int_0^\pi \frac{x\sin x}{1+\cos^2 x}dx = \frac{\pi}{2}\int_0^\pi \frac{\sin x}{1+\cos^2 x}dx = -\frac{\pi}{2}\int_0^\pi \frac{1}{1+\cos^2 x}d(\cos x) = -\frac{\pi}{2}\left[\tan^{-1}(\cos x)\right]_0^\pi$

$$= -\frac{\pi}{2}\left[\tan^{-1}(-1) - \tan^{-1}1\right] = -\frac{\pi}{2}\left[-\frac{\pi}{4} - \frac{\pi}{4}\right] = \frac{\pi^2}{4}$$

(c) $g(x) = \frac{\sqrt{1+\cos x}}{\sqrt{1+\cos x}+\sqrt{1-\cos x}} = \frac{\sqrt{1+\cos x}\left(\sqrt{1+\cos x}-\sqrt{1-\cos x}\right)}{(1+\cos x)-(1-\cos x)} = \frac{1+\cos x - \sin x}{2\cos x}$, $x \in [0, \pi]$.

$\therefore$ $g'(x) = \frac{1}{2}\frac{(\cos x)(-\sin x - \cos x) - (1+\cos x - \sin x)(-\sin x)}{(\cos x)^2} = \frac{1}{2}\frac{-\cos^2 x + \sin x - \sin^2 x}{(1-\sin x)(1+\sin x)}$

$$= \frac{1}{2}\frac{\sin x - 1}{(1-\sin x)(1+\sin x)} = -\frac{1}{2(1+\sin x)}$$

Using integration by parts, we have

$$\int_0^\pi g(x)\,dx = xg(x)\Big|_0^\pi - \int_0^\pi xg'(x)\,dx = [\pi g(\pi) - 0g(0)] + \int_0^\pi \frac{x}{2(1+\sin x)}dx = 0 + \frac{\pi}{2}\int_0^\pi \frac{1}{2(1+\sin x)}dx$$

$$= -\frac{\pi}{2}\int_0^\pi g'(x)dx = -\frac{\pi}{2}g(x)\Big|_0^\pi = -\frac{\pi}{2}[g(\pi) - g(0)] = -\frac{\pi}{2}[0-1] = \frac{\pi}{2}$$

2. $\frac{d^n}{dx^n}\left(\frac{1}{x}\right) = (-1)^n \frac{n!}{x^{n+1}}$

$$\frac{d^n}{dx^n}\left(\frac{y}{x}\right) = \sum_{k=0}^{n}\left(\frac{1}{x}\right)^{(n-k)} y^{(k)} = \sum_{k=0}^{n}(-1)^k \frac{k!}{x^{k+1}}y^{(k)}$$

$$= (-1)^n \frac{n!}{x^{n+1}}\left(y - x\frac{dy}{dx} + \frac{x^2}{2!}\frac{d^2y}{dx^2} - \frac{x^3}{3!}\frac{d^3y}{dx^3} + \ldots + (-1)^n\frac{x^n}{n!}\frac{d^n y}{dx^n}\right)$$

$$\frac{d^n}{dx^n}\left(\frac{e^{-x}}{x}\right) = (-1)^n \frac{n!}{x^{n+1}}\left(e^{-x} + xe^{-x} + \frac{x^2}{2!}e^{-x} + \frac{x^3}{3!}e^{-x} + \ldots + \frac{x^n}{n!}e^{-x}\right) \quad \ldots. \quad (1)$$

Since $I_n = \int_0^x t^n e^{-t}dt = -\int_0^x t^n de^{-t} = -t^n e^{-t}\Big|_0^x + n\int_0^x t^{n-1}e^{-t}dt = -x^n e^{-x} + n\int_0^x t^{n-1}e^{-t}dt = -x^n e^{-x} + nI_{n-1}$

$$I_n = \int_0^x t^n e^{-t}dt = -x^n e^{-x} + nI_{n-1} = -x^n e^{-x} + n\left(-x^{n-1}e^{-x} + (n-1)I_{n-2}\right) = -x^n e^{-x} - nx^{n-1}e^{-x} + n(n-1)I_{n-2} = \ldots$$

$$= n! - (-1)^n \frac{n!}{x^{n+1}}\left(e^{-x} + xe^{-x} + \frac{x^2}{2!}e^{-x} + \frac{x^3}{3!}e^{-x} + \ldots + \frac{x^n}{n!}e^{-x}\right)$$, since $I_0 = -e^{-x} + 1$

$$\frac{(-1)^n}{x^{n+1}}\left(n! - \int_0^x t^n e^{-t}dt\right) = (-1)^n \frac{n!}{x^{n+1}}\left(e^{-x} + xe^{-x} + \frac{x^2}{2!}e^{-x} + \frac{x^3}{3!}e^{-x} + \ldots + \frac{x^n}{n!}e^{-x}\right) \quad \ldots. \quad (2)$$

Combine (1) and (2), result follows.

3. **(a)** Consider the equation:

$x^{2n-1} + x^{2n-2} + \ldots + x + 1 = 0 \quad \Leftrightarrow \quad \frac{x^{2n}-1}{x-1} = 0 \quad , \quad x \neq 1 . \Leftrightarrow \quad x^{2n} - 1 = 0 , \quad x \neq 1 .$

Now, $x^{2n} = 1 = \cos 2k\pi + i \sin 2k\pi$, $\quad \therefore \quad x = \cos\left(\frac{2k\pi}{2n}\right) + i\sin\left(\frac{2k\pi}{2n}\right)$, $\quad k = 1, 2, \ldots,$ $(2n - 1)$.

Note that $k = 0$ case is excluded because $x \neq 1$.

Notice that $\cos\left(\frac{2(2n-k)\pi}{2n}\right) + i\sin\left(\frac{2(2n-k)\pi}{2n}\right) = \cos\frac{2k\pi}{2n} - i\sin\frac{2k\pi}{2n}$

$x^{2n-1} + x^{2n-2} + \ldots + x + 1$

$$= \prod_{k=1}^{2n-1}\left[x - \left(\cos\frac{2k\pi}{2n} + i\sin\frac{2k\pi}{2n}\right)\right] = (x+1)\prod_{k=1}^{n-1}\left[x - \left(\cos\frac{2k\pi}{2n} + i\sin\frac{2k\pi}{2n}\right)\right]\left[x - \left(\cos\frac{2k\pi}{2n} - i\sin\frac{2k\pi}{2n}\right)\right]$$

$$= (x+1)\prod_{k=1}^{n-1}\left[x^2 - 2x\cos\frac{k\pi}{n} + 1\right]$$

Put $x = 1$, we have $2n = 2\prod_{k=1}^{n-1} 2\left[1 - \cos\frac{k\pi}{n}\right] = 2\prod_{k=1}^{n-1} 4\sin^2\frac{k\pi}{2n}$

$$n = 4^{n-1}\prod_{k=1}^{n-1}\sin^2\frac{k\pi}{2n} = \left[2^{n-1}\prod_{k=1}^{n-1}\sin\frac{k\pi}{2n}\right]^2 \Rightarrow \sqrt{n} = 2^{n-1}\prod_{k=1}^{n-1}\sin\frac{k\pi}{2n} \Rightarrow \prod_{k=1}^{n-1}\sin\frac{k\pi}{2n} = \frac{\sqrt{n}}{2^{n-1}} \quad \ldots . \quad (1)$$

$$\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\ldots\sin\frac{(n-1)\pi}{n} = \prod_{k=1}^{n-1}\sin\frac{k\pi}{n} = \prod_{k=1}^{n-1} 2\sin\frac{k\pi}{2n}\cos\frac{k\pi}{n} = 2^{n-1}\prod_{k=1}^{n-1}\sin\frac{k\pi}{2n}\prod_{k=1}^{n-1}\sin\left(\frac{\pi}{2} - \frac{k\pi}{n}\right)$$

$$= 2^{n-1}\prod_{k=1}^{n-1}\sin\frac{k\pi}{2n}\prod_{k=1}^{n-1}\sin\left(\frac{(n-k)\pi}{2n}\right) = 2^{n-1}\left[\prod_{k=1}^{n-1}\sin\frac{k\pi}{2n}\right]^2 = 2^{n-1}\left[\frac{\sqrt{n}}{2^{n-1}}\right]^2 = \frac{n}{2^{n-1}} \quad \text{,by (1).}$$

(b) $\int_0^{\pi/2} \ln\sin x\,dx = \lim_{n\to\infty}\frac{\pi}{2n}\sum_{k=1}^{n-1} \ln\sin\frac{k\pi}{2n} = \lim_{n\to\infty}\frac{\pi}{2n}\ln\prod_{k=1}^{n-1}\sin\frac{k\pi}{2n} = \lim_{n\to\infty}\frac{\pi}{2n}\ln\left(\frac{\sqrt{n}}{2^{n-1}}\right)$, by (1)

$$= \lim_{n\to\infty}\frac{\pi}{2n}\ln\left(\frac{\sqrt{n}}{2^{n-1}}\right) = \lim_{n\to\infty}\frac{\pi}{4n}\ln\left(\frac{n}{2^{2n-2}}\right) = \frac{\pi}{4}\lim_{x\to\infty}\frac{\ln n - (2n-2)\ln 2}{n} = \frac{\pi}{4}\lim_{x\to\infty}\frac{\frac{d}{dx}(\ln x - (2x-2)\ln 2)}{\frac{d}{dx}x} \text{, LHR}$$

$$= \frac{\pi}{4}\ln\lim_{x\to\infty}\left(\frac{1}{x} - 2\ln 2\right) = -\frac{\pi}{2}\ln 2$$

4. **(a)** Consider the equation $a^{2n} - 1 = 0$, the roots are

$$a = (\text{cis } 0)^{1/2n} = (\text{cis } 2k\pi)^{1/2n} = \text{cis}\frac{2k\pi}{2n} \quad , \quad k = 0, 1, \ldots, 2n-1.$$

$$= 1, -1 \text{ or } \text{cis}\left(\pm\frac{k\pi}{n}\right), \quad k = 1, \ldots, n-1.$$

$$\therefore \quad a^{2n} - 1 = (a-1)(a+1)\prod_{k=1}^{n-1}\left[a - \text{cis}\frac{k\pi}{n}\right]\left[a - \text{cis}\left(-\frac{k\pi}{n}\right)\right] = (a^2 - 1)\prod_{r=1}^{n-1}\left(1 - 2a\cos\frac{r\pi}{n} + a^2\right)$$

(b) If $a^2 < 1$,

$$\int_0^{\pi} \ln(1 - 2a\cos x + a^2)dx = \lim_{n\to\infty}\frac{\pi}{n}\sum_{r=1}^{n} \ln\left(1 - 2a\cos\frac{r\pi}{n} + a^2\right), \quad \text{where } x_r = \frac{r\pi}{n}, \Delta x_r = \frac{\pi}{n}$$

$$= \lim_{n\to\infty}\frac{\pi}{n}\ln\prod_{r=1}^{n=1}\left(1-2a\cos\frac{r\pi}{n}+a^2\right) = \lim_{n\to\infty}\frac{\pi}{n}\ln\frac{1-a^{2n}}{1-a^2} = \lim_{n\to\infty}\frac{\pi}{n}\ln\left(\lim_{n\to\infty}\frac{1-a^{2n}}{1-a^2}\right) = \lim_{n\to\infty}\frac{\pi}{n}\times\ln\left(\frac{1}{1-a^2}\right) = 0$$

If $a^2 > 1$,

$$I = \int_0^{\pi} \ln(1-2a\cos x + a^2)dx = \lim_{n\to\infty}\frac{\pi}{n}\ln\left(\frac{1-a^{2n}}{1-a^2}\right)$$

By L' hospital rule, $I = \pi\lim_{n\to\infty}\dfrac{\frac{1-a^2}{1-a^{2n}}\left(\frac{1}{1-a^2}\right)(-a^{2n})2\ln|a|}{1} = 2\pi\ln|a| = \pi\ln a^2$

5. **(a)** **(i)** $I = \int x f''(x)\,dx$ Let $u = x$, $dv = f''(x)\,dx$, then $du = dx$, $v = f'(x)$

Integration by parts gives $I = x f'(x) - \int f'(x)\,dx = x f'(x) - f(x) + C$

(ii) $\int f'(2x)\,dx = \frac{1}{2}\int f'(2x)\,d(2x) = \frac{1}{2}f(2x) + C$

(b) **(i)** $f'(x^2) = \frac{1}{x} \Rightarrow \frac{d}{dx^2}f(x^2) = \frac{1}{x} \Rightarrow \frac{d}{dx}f(x^2)\Big/\frac{dx^2}{dx} = \frac{1}{x} \Rightarrow \frac{d}{dx}f(x^2) = 2 \Rightarrow f(x^2) = 2x + C$

$\Rightarrow f(x) = 2\sqrt{x} + C$ $(x > 0)$

(ii) $f'(\sin^2 x) = \cos^2 x$ $\Rightarrow \frac{d}{dx}f(\sin^2 x)\Big/\frac{d\sin^2 x}{dx} = \cos^2 x \Rightarrow \frac{d}{dx}f(\sin^2 x) = 2\sin x\cos^3 x$

$\Rightarrow f(\sin^2 x) = \int 2\sin x\cos^3 x\,dx = -2\int\cos^3 x\,d\cos x = -\frac{\cos^4 x}{2} + C = -\frac{(1-\sin^2 x)^2}{2} + C$

$\Rightarrow f(x) = -\frac{(1-x)^2}{2} + C$

(iii) For $0 \le x \le 1$,

$f'(\ln x) = 1 \Rightarrow \frac{d}{dx}f(\ln x)\Big/\frac{d}{dx}\ln x = 1 \Rightarrow \frac{d}{dx}f(\ln x) = \frac{1}{x} \Rightarrow f(\ln x) = -\frac{1}{x^2} + C_1 \Rightarrow f(x) = -\frac{1}{e^{2x}} + C_1$

For $1 < x < +\infty$,

$f'(\ln x) = x \Rightarrow \frac{d}{dx}f(\ln x)\Big/\frac{d}{dx}\ln x = x \Rightarrow \frac{d}{dx}f(\ln x) = 1 \Rightarrow f(\ln x) = x + C_2 \Rightarrow f(x) = e^x + C_2$

$f(0) = 0 \Rightarrow C_1 = 1.$

Since f has a derivative at $x = 1$, $-\frac{1}{e^{2x}} + C_1 = e^x + C_2$ at $x = 1$.

$-\frac{1}{e^{2(1)}} + 1 = e^1 + C_2 \Rightarrow C_2 = 1 - e - \frac{1}{e^2}$ $\therefore$ $f(x) = \begin{cases} -\frac{1}{e^{2x}} + 1 & , 0 \le x \le 1 \\ e^x + 1 - e - \frac{1}{e} & , 1 < x < +\infty \end{cases}$

6. **(a)** $I = \int_4^9 \frac{dx}{(2+\sqrt{x})^2} = \frac{1}{(2+\sqrt{\xi})^2}\int_4^9 dx = \frac{5}{(2+\sqrt{\xi})^2}$, where $4 \le \xi \le 9$.

$\therefore$ Max (I) = $\frac{5}{(2+\sqrt{4})^2} = \frac{5}{16}$ and Min (I) = $\frac{5}{(2+\sqrt{9})^2} = \frac{5}{25} = 5$.

$$\therefore \quad 5 < \int_4^9 \frac{dx}{\left(2+\sqrt{x}\right)^2} < \frac{5}{16}$$

(b) $I = \int_0^{\pi/4} \sqrt{1+\sin^4\theta}\,d\theta = \sqrt{1+\sin^4\xi}\int_0^{\pi/4} d\theta$, where $0 \le \xi \le \frac{\pi}{4}$.

$$\therefore \quad \text{Max (I)} = \sqrt{1+\sin^4\frac{\pi}{4}} = \sqrt{1+\left(\frac{1}{\sqrt{2}}\right)^4} = \sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{2} \qquad \text{and} \qquad \text{Min (I)} = \sqrt{1+\sin^4 0} = 1$$

$$\therefore \quad 1 < \int_0^{\pi/4} \sqrt{1+\sin^4\theta}\,d\theta < \frac{\sqrt{5}}{2} \ .$$

(c) $I = \int_{\pi/3}^{\pi} \frac{x dx}{1+\cos^2 x} = \frac{1}{1+\cos^2\xi}\int_{\pi/3}^{\pi} x dx = \frac{1}{1+\cos^2\xi}\left[\frac{x^2}{2}\right]_{\pi/3}^{\pi} = \frac{4\pi^2}{9\left(1+\cos^2\xi\right)}$ where $\frac{\pi}{3} \le \xi \le \pi$.

$$\therefore \quad \text{Max (I)} = \frac{4\pi^2}{9\left(1+\cos^2\frac{\pi}{2}\right)} = \frac{4\pi^2}{9} \qquad \text{and} \qquad \text{Min (I)} = \frac{4\pi^2}{9\left(1+\cos^2\pi\right)} = \frac{2\pi^2}{9}$$

$$\therefore \quad \frac{2\pi^2}{9} < \int_{\pi/3}^{\pi} \frac{x dx}{1+\cos^2 x} < \frac{4\pi^2}{9}$$

(d) $I = \int_0^{\pi/2} x\sqrt{\sin x}\,dx = \sqrt{\sin\xi}\int_0^{\pi/2} x dx = \sqrt{\sin\xi}\left[\frac{x^2}{2}\right]_0^{\pi/2} = \frac{\pi^2}{8}\sqrt{\sin\xi}$ where $0 \le \xi \le \frac{\pi}{2}$.

$$\therefore \quad \text{Max (I)} = \frac{\pi^2}{8}\sqrt{\sin\frac{\pi}{2}} = \frac{\pi^2}{8} \qquad \text{and} \qquad \text{Min (I)} = \frac{\pi^2}{8}\sqrt{\sin 0} = 0$$

$$\therefore \quad 0 < \int_0^{\pi/2} x\sqrt{\sin x}\,dx < \frac{\pi^2}{8}$$

(e) $I = \int_0^1 x^{\frac{1}{4}} e^{-x} dx = \xi^{\frac{1}{4}}\int_0^1 e^{-x} dx = -\xi^{\frac{1}{4}}\left[e^{-x}\right]_0^1 = \xi^{\frac{1}{4}}\left[1-\frac{1}{e}\right]$, where $0 \le \xi \le 1$.

$$\therefore \quad \text{Max (I)} = 1^{\frac{1}{4}}\left[1-\frac{1}{e}\right] = 1-\frac{1}{e} \qquad \text{and} \qquad \text{Min (I)} = 0^{\frac{1}{4}}\left[1-\frac{1}{e}\right] = 0$$

7. **(a)** Put $y = 1 - x$, $B(p, q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx = \int_1^0 (1-y)^{p-1}y^{q-1}(-dy) = \int_0^1 y^{q-1}(1-y)^{p-1}dy = B(q, p)$

(b) **(i)** Using integration by parts,

$$B(p, q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx = \frac{1}{p}\int_0^1 (1-x)^{q-1}d\left(x^p\right) = \frac{1}{p}\left[x^p(1-x)^{q-1}\right]_0^1 - \frac{1}{p}\int_0^1 x^p d(1-x)^{q-1}$$

$$= 0 - \frac{q-1}{p}\int_0^1 x^p(1-x)^{q-2}(-1)dx = \frac{q-1}{p}\int_0^1 x^{p-1}(1-x)^{q-2}\left[1-(1-x)\right]dx$$

$$= \frac{q-1}{p}\int_0^1 x^{p-1}(1-x)^{q-2}dx - \frac{q-1}{p}\int_0^1 x^{p-1}(1-x)^{q-1}dx$$

$$= \frac{q-1}{p}B(p, q-1) - \frac{q-1}{p}B(p, q)$$

Solving B(p, q), we get $B(p, q) = \frac{q-1}{p+q-1}B(p, q-1)$ (1)

(ii) $B(p, q) = B(q, p) = \frac{p-1}{p+q-1}B(q, p-1)$, by (1) $\quad \therefore \quad B(p, q) = \frac{p-1}{p+q-1}B(p-1, q)$, by (a).

(c) Put $y = x^m$, $dy = m\,x^{m-1}dx \quad \Rightarrow dx = \frac{dy}{mx^{m-1}} = \frac{dy}{my^{\frac{m-1}{m}}}$

$$\int_0^1 x^{p-1}(1-x^m)^{q-1}dx = \int_0^1 y^{\frac{p-1}{m}}(1-y)^{q-1}\frac{dy}{my^{\frac{m-1}{m}}} = \frac{1}{m}\int_0^1 y^{\frac{p}{m}-1}(1-y)^{q-1}dy = \frac{1}{m}B\left(\frac{p}{m}, q\right)$$

8. **(b)** Integrating (*), we get

$$\sum_{k=0}^{m+n} C_k^{m+n} u^k \int_a^b (x-a)^k (x-b)^{m+n-k} dx = \int_a^b [(1+u)x - (au+b)]^{m+n} dx$$

$$= \frac{1}{1+u}\frac{1}{m+n+1}\left\{[(b-a)u]^{m+n+1} - (a-b)^{m+n+1}\right\} = \frac{b-a}{m+n+1}\sum_{k=0}^{m+n}(-1)^{m+n-k}(b-a)^{m+n}u^k$$

Comparing the coefficient of u^k, $\int_a^b (x-a)^m (x-b)^n\, dx = \frac{(-1)^n m!\,n!}{(m+n+1)!}(b-a)^{m+n+1}$

9. Let $I_n = \int_{n\pi}^{(n+1)\pi} e^{-ax} f(x)\,dx$

Let $y = x - n\pi$, $x = y + n\pi$, $dx = dy$

$\therefore \quad I_n = \int_0^\pi e^{-a(y+n\pi)} f(y+n\pi)\,dy = e^{-an\pi}\int_0^\pi e^{-ay} f(y)\,dy$, as f is periodic with period π, $f(y + n\pi) = f(y)$

$$= e^{-an\pi}\int_0^\pi e^{-ax} f(x)\,dx$$

$\int_0^\infty e^{-ax} f(x)\,dx = \sum_0^\infty I_n = \sum_0^\infty e^{-an\pi}\int_0^\pi e^{-ax} f(x)\,dx = \left(\int_0^\pi e^{-ax} f(x)\,dx\right)\sum_0^\infty e^{-an\pi}$, infinite G.P.

$$= \frac{1}{1-e^{-a\pi}}\int_0^\pi e^{-ax} f(x)\,dx = \frac{e^{a\pi}}{e^{a\pi}-1}\int_0^\pi e^{-ax} f(x)\,dx$$

10. $$\frac{dy}{dx} = \frac{\left(b + \int_0^x f(t)\,dt\right)\frac{d}{dx}\left(a + \int_0^x t\,f(t)\,dt\right) - \left(a + \int_0^x t\,f(t)\,dt\right)\frac{d}{dx}\left(b + \int_0^x f(t)\,dt\right)}{\left(b + \int_0^x f(t)\,dt\right)^2}$$

$= \frac{\left(b + \int_0^x f(t)\,dt\right)[xf(x)] - \left(a + \int_0^x t\,f(t)\,dt\right)[f(x)]}{\left(b + \int_0^x f(t)\,dt\right)^2}$, by Fundamental theorem of integral calculus.

$\left.\frac{dy}{dx}\right|_{x=X} = 0 \Rightarrow \left(b + \int_0^X f(t)\,dt\right)[Xf(X)] = \left(a + \int_0^X t\,f(t)\,dt\right)[f(X)] \Rightarrow \left(b + \int_0^X f(t)\,dt\right)X = \left(a + \int_0^X t\,f(t)\,dt\right)$, $f(X) > 0$.

$$X = \frac{a + \int_0^X t\,f(t)\,dt}{b + \int_0^X f(t)\,dt} = y|_{x=X} = Y$$

11. $\int_a^b [f_1(x) - tf_2(x)]^2 dx \ge 0 \Leftrightarrow t^2\int_a^b [f_2(x)]^2 dx - 2t\int_a^b f_1(x)f_2(x)\,dx + \int_a^b [f_1(x)]^2 dx \ge 0 \qquad \forall\, t \in \mathbb{R}$.

$$\Rightarrow \quad \Delta \le 0 \qquad \Rightarrow \quad \int_a^b f_1(x)f_2(x)\,dx \le \sqrt{\int_a^b [f_1(x)]^2 dx}\sqrt{\int_a^b [f_2(x)]^2 dx}$$

12. $x^2 - x = \left(x - \frac{1}{2}\right)^2 - \frac{1}{4} \Rightarrow -\frac{1}{4} \le x^2 - x \le 2 \qquad \forall x \in [0,2]$

$$\Rightarrow e^{-\frac{1}{4}} \le e^{x^2-x} \le e^2 \qquad \Rightarrow \int_0^2 e^{-\frac{1}{4}} dx \le \int_0^2 e^{x^2-x}\,dx \le \int_0^2 e^2 dx \qquad \Rightarrow 2e^{-\frac{1}{4}} \le \int_0^2 e^{x^2-x}\,dx \le 2e^2$$

13. **(a)** $F(n\pi) = G(n\pi) = 0$ since $\sin(n\pi) = 0$. F and G are periodic since sin x and cos x are periodic.

F and G are odd since sin x is odd and cos x is even.

(b) $F(\alpha) = \int_{-1}^{1} \frac{\sin\alpha}{x^2 + 2x\cos\alpha + 1}dx = \int_{-1}^{1} \frac{\sin\alpha}{x^2 + 2x\cos\alpha + \cos^2\alpha + \sin^2\alpha}dx$

$$= \int_{-1}^{1} \frac{\sin\alpha}{(x+\cos\alpha)^2 + \sin^2\alpha}dx = \left[\tan^{-1}\frac{x+\cos\alpha}{\sin\alpha}\right]_{-1}^{1} = \tan^{-1}\left(\frac{1+\cos\alpha}{\sin\alpha}\right) - \tan^{-1}\left(\frac{-1+\cos\alpha}{\sin\alpha}\right)$$

$$= \tan^{-1}\left(\frac{2\cos^2\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}\right) - \tan^{-1}\left(\frac{-2\sin^2\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}\right) = \tan^{-1}\left(\cot\frac{\alpha}{2}\right) - \tan^{-1}\left(-\tan\frac{\alpha}{2}\right)$$

$$= \tan^{-1}\left(\cot\frac{\alpha}{2}\right) + \tan^{-1}\left(\tan\frac{\alpha}{2}\right)$$

For $0 < \alpha < \pi$, $F(\alpha) = \tan^{-1}\left[\tan\left(\frac{\pi}{2} - \frac{\alpha}{2}\right)\right] + \tan^{-1}\left(\tan\frac{\alpha}{2}\right) = \left(\frac{\pi}{2} - \frac{\alpha}{2}\right) + \frac{\alpha}{2} = \frac{\pi}{2}$

For $-\pi < \alpha < 0$, $-\frac{\pi}{2} < \frac{\alpha}{2} < 0$, $F(\alpha) = \tan^{-1}\left[\tan\left(-\frac{\pi}{2} - \frac{\alpha}{2}\right)\right] + \tan^{-1}\left(\tan\frac{\alpha}{2}\right) = \left(-\frac{\alpha}{2} - \frac{\pi}{2}\right) + \frac{\alpha}{2} = -\frac{\pi}{2}$

(b) $G(\alpha) = \int_0^1 \frac{\sin\alpha}{x^2 + 2x\cos\alpha + 1}dx = \left[\tan^{-1}\frac{x+\cos\alpha}{\sin\alpha}\right]_0^1 = \tan^{-1}\left(\frac{1+\cos\alpha}{\sin\alpha}\right) - \tan^{-1}\left(\frac{\cos\alpha}{\sin\alpha}\right)$

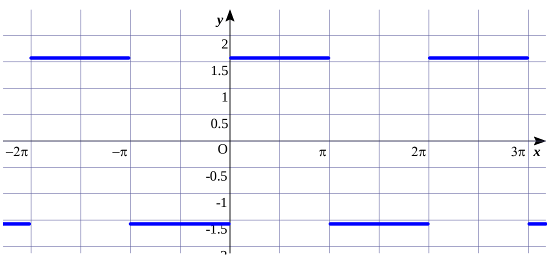
$$= \tan^{-1}\left(\frac{2\cos^2\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}\right) - \tan^{-1}(\cot\alpha) = \tan^{-1}\left(\cot\frac{\alpha}{2}\right) - \tan^{-1}(\cot\alpha)$$

For $0 < \alpha < \pi$, $G(\alpha) = \tan^{-1}\left(\tan\left(\frac{\pi}{2} - \frac{\alpha}{2}\right)\right) - \tan^{-1}\left(\tan\left(-\frac{\pi}{2} - \alpha\right)\right) = \left(\frac{\pi}{2} - \frac{\alpha}{2}\right) - \left(\frac{\pi}{2} - \alpha\right) = \frac{\alpha}{2}$

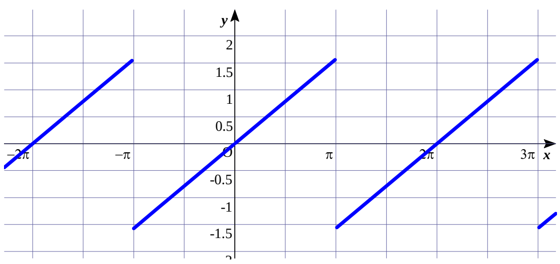
For $-\pi < \alpha < 0$, $-\frac{\pi}{2} < \frac{\alpha}{2} < 0$,

$$G(\alpha) = \tan^{-1}\left(\tan\left(-\frac{\pi}{2} - \frac{\alpha}{2}\right)\right) - \tan^{-1}\left(\tan\left(-\frac{\pi}{2} - \alpha\right)\right) = \left(-\frac{\pi}{2} - \frac{\alpha}{2}\right) - \left(-\frac{\pi}{2} - \alpha\right) = \frac{\alpha}{2}$$

(c) $y = F(\alpha)$:



$y = G(\alpha)$:



14. Let $t = u^n$, When $t = 1, u = 1$. When $t = x^n$, $u = x$. Also, $dt = nu^{n-1}\,du$.

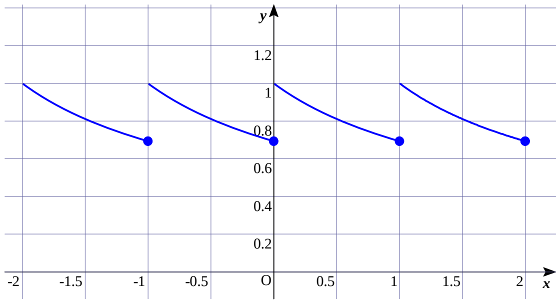
$$\ln x^n = \int_1^x \frac{nu^{n-1}du}{u^n} = n\int_1^x \frac{du}{u} = n\ln u\Big|_1^x = n\ln x$$

Since $\frac{1}{t}$ is a decreasing function , $\forall\, t \in [1, 1 + x]$, we have $\frac{1}{1+x} < \frac{1}{t} < \frac{1}{1} = 1$, where $x > 0$.

From the graph (omitted), we get the area on integrating,

$$\int_1^{1+x} \frac{1}{1+x}\,dt < \int_1^{1+x} \frac{1}{t}\,dt < \int_1^{1+x} dt \quad \Rightarrow \frac{x}{1+x} < \ln(1+x) < x\,, \quad \text{where} \quad x > 0\,.$$

$$\Rightarrow \frac{1}{1+x} < \frac{\ln(1+x)}{x} < 1$$



Taking limit as $x \to 0$, and using the Sandwich theorem,

$$\lim_{x\to 0} \frac{1}{1+x} = 1, \qquad \Rightarrow \qquad \lim_{x\to 0} \frac{\ln(1+x)}{x} = 1$$

15. **(a)** $y = \frac{1}{x} \quad \Rightarrow y' = -\frac{1}{x^2} < 0 \Rightarrow$ y is increasing for all non-zero x .

$$0 < \frac{1}{r} < \frac{1}{x} < \frac{1}{r-1}, \qquad \forall x, \quad 0 < r-1 < x < r$$

$$\int_{r-1}^r \frac{1}{r}\,dx < \int_{r-1}^r \frac{1}{x}\,dx < \int_{r-1}^r \frac{1}{r-1}\,dx \quad \Rightarrow \quad \frac{1}{r} < \int_{r-1}^r \frac{dx}{x} < \frac{1}{r-1}$$

$$\sum_{r=2}^n \frac{1}{r} < \sum_{r=2}^n \int_{r-1}^r \frac{dx}{x} < \sum_{r=2}^n \frac{1}{r-1} \quad \Rightarrow \quad \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} < \int_1^n \frac{dx}{x} < 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-1}$$

(b) $0 < \frac{1}{r} < \int_{r-1}^r \frac{dx}{x} < \frac{1}{r-1} \Rightarrow 0 < \frac{1}{r} < \ln\frac{r}{r-1} < \frac{1}{r-1} \quad \Rightarrow 0 < \ln\frac{r}{r-1} - \frac{1}{r} < \frac{1}{r-1} - \frac{1}{r}$

$$\therefore \quad 0 < \sum_{r=n+1}^{2n} \ln\frac{r}{r-1} - \sum_{r=n+1}^{2n} \frac{1}{r} < \sum_{r=n+1}^{2n} \left(\frac{1}{r-1} - \frac{1}{r}\right) \Rightarrow 0 < \ln\frac{2n}{n} - \sum_{r=n+1}^{2n} \frac{1}{r} < \frac{1}{n} - \frac{1}{2n} \Rightarrow 0 < \ln 2 - \sum_{r=n+1}^{2n} \frac{1}{r} < \frac{1}{2n}$$

(c) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{1}{2n-1} - \frac{1}{2n} = \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{2n-1} + \frac{1}{2n}\right) - 2\left(\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2n}\right)$

$$= 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{2n} - \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right) = a_{2n} - a_n$$

$$a_{2n} - a_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{2n} - \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right) = \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n} = \sum_{r=n+1}^{2n} \frac{1}{r}$$

$$\therefore \quad 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{1}{2n-1} - \frac{1}{2n} = a_{2n} - a_n = \sum_{r=n+1}^{2n} \frac{1}{r} \qquad (1)$$

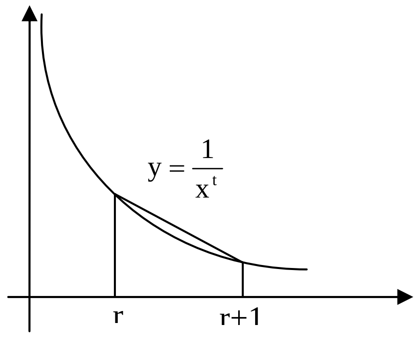
Now, from **(b),** $0 < \ln 2 - \sum_{r=n+1}^{2n} \frac{1}{r} < \frac{1}{2n} \Rightarrow 0 \le \ln 2 - \lim_{n\to\infty} \sum_{r=n+1}^{2n} \frac{1}{r} \le \lim_{n\to\infty} \frac{1}{2n} = 0$

By Sandwich theorem, $\lim_{n\to\infty} \sum_{r=n+1}^{2n} \frac{1}{r} = \ln 2$. By (1), $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots = \lim_{n\to\infty} \sum_{r=n+1}^{2n} \frac{1}{r} = \ln 2$

16. **(a)** $y=\frac{1}{x^t}\Rightarrow\frac{dy}{dx}=-\frac{t}{x^{t+1}}\Rightarrow\frac{d^2y}{dx^2}=\frac{t(t+1)}{x^{t+2}}>0$, for all $t>0$, $x>0$.

$\therefore$ The gradient of the graph of $y=\frac{1}{x^t}$ is negative and increases steadily as x increases.

(1) $\int_r^{r+1}\frac{1}{x^t}dx$

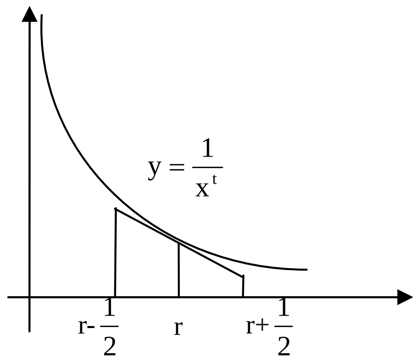


= area under the curve $y=\frac{1}{x^t}$

< area of the trapezium

$=\frac{1}{2}\left[\frac{1}{r^t}+\frac{1}{(r+1)^t}\right]$

(2) $\int_{r-\frac{1}{2}}^{r+\frac{1}{2}}\frac{1}{x^t}dx$ which hold for $t>0$, $r\ge 1$.



> area of the trapezium

= rectangle with height $\frac{1}{r^t}$ and width $1=\frac{1}{r^t}$

(b) From (1),

$\sum_{r=1}^{n-1}\int_r^{r+1}\frac{1}{x^t}dx<\frac{1}{2}\sum_{r=1}^{n-1}\left[\frac{1}{r^t}+\frac{1}{(r+1)^t}\right]$

$\int_1^n\frac{1}{x^t}dx<1+\frac{1}{2^t}+\frac{1}{3^t}+\ldots+\frac{1}{n^t}-\frac{1}{2}\left(1+\frac{1}{n^t}\right)$

$\therefore\quad\frac{1}{2}+\frac{1}{2n^t}<G_n(t)=1+\frac{1}{2^t}+\frac{1}{3^t}+\ldots+\frac{1}{n^t}-\int_1^n\frac{1}{x^t}dx$

From (2), $\sum_{r=1}^{n}\int_{r-\frac{1}{2}}^{r+\frac{1}{2}}\frac{1}{x^t}dx>\sum_{r=1}^{n}\frac{1}{r^t}\Rightarrow\int_{\frac{1}{2}}^{n+\frac{1}{2}}\frac{1}{x^t}dx>1+\frac{1}{2^t}+\frac{1}{3^t}+\ldots+\frac{1}{n^t}$

$\therefore\quad G_n(t)=1+\frac{1}{2^t}+\frac{1}{3^t}+\ldots+\frac{1}{n^t}-\int_1^n\frac{1}{x^t}dx<\int_{\frac{1}{2}}^{n+\frac{1}{2}}\frac{1}{x^t}dx-\int_1^n\frac{1}{x^t}dx=\int_{1/2}^1\frac{1}{x^t}dx+\int_n^{n+\frac{1}{2}}\frac{1}{x^t}dx$

(c) $\int_{1/2}^1\frac{1}{x^t}dx=\int_{1/2}^1x^{-t}dx=\left[\frac{x^{-t+1}}{-t+1}\right]_{1/2}^1=\frac{1}{1-t}\left(1-\frac{1}{2^{1-t}}\right)$, if $t\ne 1$.

$\int_n^{n+\frac{1}{2}}\frac{1}{x^t}dx<\int_n^{n+\frac{1}{2}}\frac{1}{n^t}dx=\frac{1}{2n^t}$, since the integrand is a decreasing function.

(d) From (b), (c), $\frac{1}{2}+\frac{1}{2n^t}<G_n(t)<\int_{1/2}^1\frac{1}{x^t}dx+\int_n^{n+\frac{1}{2}}\frac{1}{x^t}dx<\frac{1}{1-t}\left(1-\frac{1}{2^{1-t}}\right)+\frac{1}{2n^t}$

$\therefore\quad\frac{1}{2}\le\lim_{n\to\infty}G_n(t)\le\left[\frac{1}{1-t}\left(1-\frac{1}{2^{1-t}}\right)\right]$ since $\lim_{n\to\infty}\frac{1}{2n^t}=0$

and by Sandwich theorem, $\lim_{t\to 0^+}\left[\lim_{n\to\infty}G_n(t)\right]=\frac{1}{2}$ since $\lim_{n\to\infty}\left[\frac{1}{1-t}\left(1-\frac{1}{2^{1-t}}\right)\right]=\frac{1}{2}$.

$$\lim_{t\to0^+} G_n(t) = \lim_{t\to0^+}\left[1+\frac{1}{2^t}+\frac{1}{3^t}+\ldots+\frac{1}{n^t}-\int_1^n \frac{1}{x^t}dx\right] = n-\int_1^n dx = n-[n-1]=1 \Rightarrow \lim_{n\to\infty}\left[\lim_{t\to0^+} G_n(t)\right] = 1$$

17. Consider $g(\lambda) = f(\lambda a + (1-\lambda)b) - \lambda f(a) - (1-\lambda) f(b)$ as a function of λ, for all $1 \ge \lambda \ge 0$

$g'(\lambda) = (a-b) f'(\lambda a + (1-\lambda)b) - f(a) + f(b)$

$g''(\lambda) = (a-b)^2 f''(\lambda a + (1-\lambda)b) < 0$, since $f''(x) < 0$ for all $x > 0$ …. (1)

Since $g(0) = g(1) = 0$, by Mean Value Theorem, $\exists\, \xi$, $0 \le \xi \le 1$, such that $g'(\xi) = \dfrac{g(1)-g(0)}{1-0} = \dfrac{0}{1} = 0$.

$\forall x$, $0 \le x < \xi$, by Mean Value Theorem, $\exists \xi_1$, $x \le \xi_1 < \xi$, such that $g''(\xi_1) = \dfrac{g'(\xi)-g'(x)}{\xi - x} = -\dfrac{g'(x)}{\xi - x}$.

By (1), $g''(\xi_1) < 0$ and since $\xi > x$, we have $g'(x) > 0$.

$\forall x$, $\xi < x \le 1$, by Mean Value Theorem, $\exists \xi_2$, $\xi < \xi_2 \le x$, such that $g''(\xi_2) = \dfrac{g'(x)-g'(\xi)}{x-\xi} = \dfrac{g'(x)}{x-\xi}$.

By (1), $g''(\xi_2) < 0$ and since $\xi > 0$, we have $g'(x) < 0$.

$\therefore$ There is one and only one max of $g(\lambda)$ when $\lambda = \xi$.

Since f is a twice differentiable function and is therefore continuous, g is also continuous.

Since $g(0) = g(1) = 0$ and g has only one max, we have $g(\lambda) \ge 0$ for all λ, for all $1 \ge \lambda \ge 0$

$\therefore f(\lambda a + (1-\lambda)b) - \lambda f(a) - (1-\lambda) f(b) \ge 0$ and hence $f(\lambda a + (1-\lambda)b) \ge \lambda f(a) + (1-\lambda) f(b)$

18. **(a)** $f(x) = x - \dfrac{1}{3}\left(8\sin\dfrac{x}{2} - \sin x\right)$

$$f'(x) = 1-\frac{1}{3}\left(4\cos\frac{x}{2}-\cos x\right) = 1-\frac{1}{3}\left[4\cos\frac{x}{2}-\left(2\cos^2\frac{x}{2}-1\right)\right] = \frac{2}{3}\left[\cos^2\frac{x}{2}-2\cos\frac{x}{2}+1\right]$$

$$= \frac{2}{3}\left(\cos\frac{x}{2}-1\right)^2 = \frac{2}{3}\left(2\sin^2\frac{x}{4}\right)^2 = \underline{\underline{\frac{8}{3}\sin^4\frac{x}{4}}}$$

(b) From (a), $f(x) = \dfrac{8}{3}\displaystyle\int_0^x \sin^4\frac{x}{4}dx$

For $x > 0$, $\cos\dfrac{x}{4} < 1$ and $\sin\dfrac{x}{4} < \dfrac{x}{4}$ $\quad\therefore \sin^4\dfrac{x}{4}\cos\dfrac{x}{4} < \sin^4\dfrac{x}{4} < \left(\dfrac{x}{4}\right)^4$

$$\therefore \frac{8}{3}\int_0^x \sin^4\frac{x}{4}\cos\frac{x}{4}dx < \frac{8}{3}\int_0^x \sin^4\frac{x}{4}dx < \frac{8}{3}\int_0^x \left(\frac{x}{4}\right)^4 dx$$

$$\therefore \frac{32}{15}\sin^5\frac{x}{4} < f(x) < \frac{32}{15}\left(\frac{x}{4}\right)^5$$

(c) $f\left(\dfrac{\pi}{6}\right) = \dfrac{\pi}{6}-\dfrac{1}{3}\left(8\sin\dfrac{\pi}{12}-\sin\dfrac{\pi}{6}\right) = \dfrac{\pi}{6}-\dfrac{1}{3}\left[8\sin\left(\dfrac{\pi}{4}-\dfrac{\pi}{6}\right)-\sin\dfrac{\pi}{6}\right]$

$$= \frac{\pi}{6}-\frac{1}{3}\left[8\left(\sin\frac{\pi}{4}\cos\frac{\pi}{6}-\cos\frac{\pi}{4}\sin\frac{\pi}{6}\right)-\frac{1}{2}\right] = \frac{\pi}{6}-\frac{1}{3}\left[8\left(\frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2}-\frac{1}{\sqrt{2}}\frac{1}{2}\right)-\frac{1}{2}\right]$$

$$= \frac{\pi}{6}-\frac{1}{3}\left[4\left(\sqrt{\frac{3}{2}}-\sqrt{\frac{1}{2}}\right)-\frac{1}{2}\right] = \frac{\pi}{6}-\frac{1}{3}\left[2\left(\sqrt{6}-\sqrt{2}\right)-\frac{1}{2}\right] = \underline{\underline{\frac{\pi}{6}-\frac{1}{6}\left[4\left(\sqrt{6}-\sqrt{2}\right)-1\right]}}$$

From (b), $\frac{32}{15}\sin^5\frac{\pi}{24} < f(\frac{\pi}{6}) < \frac{32}{15}\left(\frac{\pi}{24}\right)^5$

Since $\frac{32}{15}\sin^5\frac{\pi}{24} \approx 0$, $\frac{32}{15}\left(\frac{\pi}{24}\right)^5 \approx 0$, $\quad f\left(\frac{\pi}{6}\right) = \frac{\pi}{6} - \frac{1}{6}\left[4\left(\sqrt{6}-\sqrt{2}\right)-1\right] \approx 0$

$\therefore \quad \pi \approx 4(\sqrt{6}-\sqrt{2})-1$

19. Let $f(x) = \frac{(n+1+x)^{n+1}}{(n+x)^n}$, then $f'(x) = \frac{x(n+1+x)^n}{(n+x)^{n+1}} > 0$, since $x > 0$, $n \in \mathbb{N}$.

$\therefore$ f(x) is increasing.

$\therefore \quad f(x) > f(0) \quad \Rightarrow \quad f(x) = \frac{(n+1+x)^{n+1}}{(n+x)^n} > f(0) = \frac{(n+1)^{n+1}}{n^n} \Leftrightarrow \frac{(n+1+x)^{n+1}}{(n+1)^{n+1}} > \frac{(n+x)^n}{n^n}$

$$\Rightarrow \quad \left(1+\frac{x}{n}\right)^n < \left(1+\frac{x}{n+1}\right)^{n+1}$$

Let $g(x) = \frac{(n+1-x)^{n+1}}{(n-x)^n}$, where $0 < x < n$, $n \in \mathbb{N}$. $\quad g'(x) = \frac{x(1+n-x)^n}{(n-x)^{n+1}} > 0$

$\therefore \quad g(x) > g(0) \quad \Rightarrow \quad g(x) = \frac{(n+1-x)^{n+1}}{(n-x)^n} > f(0) = \frac{(n+1)^{n+1}}{n^n} \Leftrightarrow \frac{(n+1-x)^{n+1}}{(n+1)^{n+1}} > \frac{(n-x)^n}{n^n}$

$$\Rightarrow \quad \left(1-\frac{x}{n}\right)^n < \left(1-\frac{x}{n+1}\right)^{n+1}$$

20. **(a), (b)** Bookwork, Leibnitz's Theorem. omitted

(c) $\left(x^2+1\right)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - m^2y = 0 \quad \ldots\ (1)$

Differentiate n-times and using Leibnitz's theorem,

$$\left[\left(x^2+1\right)y^{(n+2)} + n(2x)y^{(n+1)} + \frac{n(n-1)}{2!}(2)y^{(n)}\right] + \left[xy^{(n+1)} + n(1)y^{(n)}\right] - m^2y^{(n)} = 0$$

$$\therefore \quad \left(x^2+1\right)y^{(n+2)} + (2n+1)xy^{(n+1)} + \left(n^2-m^2\right)y^{(n)} = 0 \quad \ldots\ (2)$$

Put n = m in (2), $\left(x^2+1\right)y^{(m+2)} + (2m+1)xy^{(m+1)} + \left(m^2-m^2\right)y^{(m)} = 0$

If $g(x) = f^{(m+1)}(x) = y^{(m)}$, then $\frac{g'(x)}{g(x)} = -\frac{(2m+1)x}{x^2+1}$.

$$\int\frac{g'(x)}{g(x)}dx = -\int\frac{(2m+1)x}{x^2+1}dx \Rightarrow \int\frac{d[g(x)]}{dx} = -\left(\frac{2m+1}{2}\right)\int\frac{d\left(x^2+1\right)}{\left(x^2+1\right)}$$

$$\Rightarrow \ln[g(x)] = -\left(\frac{2m+1}{2}\right)\ln\left(x^2+1\right) + \ln C \Rightarrow g(x) = \frac{C}{\left(x^2+1\right)^{(2m+1)/2}}$$

21. **(a)** $\int\frac{\cos\theta d\theta}{(2+\cos\theta)^n} = \int\frac{(2+\cos\theta)-2}{(2+\cos\theta)^n}d\theta = \int\frac{d\theta}{(2+\cos\theta)^{n-1}} - 2\int\frac{d\theta}{(2+\cos\theta)^n}$

$= I_{n-1} - 2I_n$

(b) $\int\frac{\cos^2\theta d\theta}{(2+\cos\theta)^n} = \int\frac{(2+\cos\theta)^2 - 4\cos\theta - 4}{(2+\cos\theta)^n}d\theta$

$$= \int \frac{d\theta}{(2+\cos\theta)^{n-2}} - 4\int \frac{\cos\theta d\theta}{(2+\cos\theta)^{n}} - 4\int \frac{d\theta}{(2+\cos\theta)^{n}}$$

$= I_{n-2} - 4(I_{n-1} - 2I_n) - 4I_n \qquad = I_{n-2} - 4I_{n-1} + 4I_n$

(c) $$\frac{d}{dx}\left[\frac{\sin\theta}{(2+\cos\theta)^{n-1}}\right] = \frac{(2+\cos\theta)^{n-1}\cos\theta - (n-1)(2+\cos\theta)^{n-2}(-\sin\theta)\sin\theta}{(2+\cos\theta)^{2n-2}}$$

$$= \frac{2\cos\theta + \cos^2\theta + (n-1)\sin^2\theta}{(2+\cos\theta)^n} = \frac{2\cos\theta + \cos^2\theta + (n-1)(1-\cos^2\theta)}{(2+\cos\theta)^n}$$

$$= \frac{2\cos\theta - (n-2)\cos^2\theta + (n-1)}{(2+\cos\theta)^n} = 2\frac{\cos\theta}{(2+\cos\theta)^n} - (n-2)\frac{\cos^2\theta}{(2+\cos\theta)^n} + (n-1)\frac{1}{(2+\cos\theta)^n}$$

Integrate the above identity, we get:

$$\frac{\sin\theta}{(2+\cos\theta)^{n-1}} = 2\int\frac{\cos\theta d\theta}{(2+\cos\theta)^n} - (n-2)\int\frac{\cos^2\theta d\theta}{(2+\cos\theta)^n} + (n-1)I_n$$

$= 2(I_{n-1} - 2I_n) - (n-2)(I_{n-2} - 4I_{n-1} + 4I_n) + (n-1)I_n$

$$3(n-1)I_n = -\frac{\sin\theta}{(2+\cos\theta)^{n-1}} - (n-2)I_{n-2} + 2(2n-3)I_{n-1}$$

$$\therefore\quad I_n = \frac{1}{3(n-1)}\left[-\frac{\sin\theta}{(2+\cos\theta)^{n-1}} - (n-2)I_{n-2} + 2(2n-3)I_{n-1}\right]$$

(d) $$\int_0^{2\pi/3}\frac{d\theta}{(2+\cos\theta)^2} = \frac{1}{3}\left[-\frac{\sin\theta}{(2+\cos\theta)^1}\Bigg|_0^{2\pi/3} + 2\int_0^{2\pi/3}\frac{d\theta}{(2+\cos\theta)}\right]$$

$$= \frac{1}{3}\left[-\frac{\sqrt{3}/2}{2-(1/2)} + 2\int_0^{\sqrt{3}}\frac{1}{2+\frac{1-t^2}{1+t^2}}\frac{2dt}{1+t^2}\right], \qquad t = \tan\frac{\theta}{2},\quad d\theta = \frac{2dt}{1+t^2},\ \cos\theta = \frac{1-t^2}{1+t^2}$$

$$= \frac{1}{3}\left[-\frac{\sqrt{3}}{3} + 4\int_0^{\sqrt{3}}\frac{dt}{3+t^2}\right] = \frac{1}{3}\left[-\frac{\sqrt{3}}{3} + \frac{4}{\sqrt{3}}\tan^{-1}\frac{t}{\sqrt{3}}\Bigg|_0^{\sqrt{3}}\right] = \frac{1}{3}\left[-\frac{\sqrt{3}}{3} + \frac{4}{\sqrt{3}}\frac{\pi}{4}\right] \qquad = \underline{\underline{\frac{\sqrt{3}}{9}(\pi-1)}}$$

22. $f(xy) = f(x) + f(y)$ …. (1)

Put $y = 1$, $f(x\times1) = f(x) + f(1) \Rightarrow f(1) = 0$ …. (2)

$$f'(x) = \lim_{h\to0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to0}\frac{f\left(x\left(1+\frac{h}{x}\right)\right)-f(x)}{h} = \lim_{h\to0}\frac{f(x)+f\left(1+\frac{h}{x}\right)-f(x)}{h}$$, by (1).

$$= \lim_{h\to0}\frac{f\left(1+\frac{h}{x}\right)}{h} = \frac{1}{x}\lim_{\frac{h}{x}\to0}\frac{f\left(1+\frac{h}{x}\right)}{\frac{h}{x}} = \frac{1}{x}\lim_{\frac{h}{x}\to0}\frac{f\left(1+\frac{h}{x}\right)-f(1)}{\frac{h}{x}}$$, by (2) and $x \neq 0$.

$$= \frac{1}{x}\lim_{h'\to0}\frac{f(1+h')-f(1)}{h'} = \frac{f'(1)}{x}$$

23. **(a)** **(i)** $h(x) = \frac{x}{\ln x} \Rightarrow h'(x) = \frac{\ln x - 1}{\ln^2 x}\begin{cases} < 0 & , x < e \\ = 0 & , x = e \\ > 0 & , x > e \end{cases}$ $\therefore$ $h(x)$ has a local minimum at $x = e$

(ii) By (i), $h(x) \ge h(e) = \frac{e}{\ln e} = e$, for all $x \in (1, \infty)$.

(b) $f(x) = \frac{x^b}{b^x} \Rightarrow f'(x) = \frac{x^{b-1}(b - x\ln b)}{b^x}$

$f(x)$ is increasing $\Leftrightarrow f'(x) > 0 \Leftrightarrow b - x \ln b > 0 \Leftrightarrow x \in \left(1, \frac{b}{\ln b}\right)$

$f(x)$ is decreasing $\Leftrightarrow f'(x) < 0 \Leftrightarrow b - x \ln b < 0 \Leftrightarrow x \in \left(\frac{b}{\ln b}, \infty\right)$

(c) Given that $1 < a < b < e$, $h(x)$ is decreasing on $(1, e)$, by (a), $h(b) > h(e)$. $\therefore$ $\frac{b}{\ln b} > \frac{e}{\ln e} = e$

By (b), $f(x)$ is increasing on $\left(1, \frac{b}{\ln b}\right)$, $f(a) < f(b)$ $\therefore$ $\frac{a^b}{b^a} < \frac{b^b}{b^b} = 1$ or $a^b < b^a$.

24. **(a)** **(i)** For any $x \ge 0$, $(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + ... + x^n > \frac{n(n-1)}{2}x^2$, $\forall n \in \mathbb{N}$.

Put $x = \sqrt[n]{n} - 1$,

$$\left(1+\sqrt[n]{n}-1\right)^n > \frac{n(n-1)}{2}\left(\sqrt[n]{n}-1\right)^2 \Rightarrow n > \frac{n(n-1)}{2}\left(\sqrt[n]{n}-1\right)^2 \Rightarrow 1 > \frac{(n-1)}{2}\left(\sqrt[n]{n}-1\right)^2$$

$$\Rightarrow \sqrt{\frac{2}{n-1}} > \sqrt[n]{n} - 1 \Rightarrow \lim_{n\to\infty}\sqrt{\frac{2}{n-1}} > \lim_{n\to\infty}\sqrt[n]{n} - 1 \Rightarrow 0 \ge \lim_{n\to\infty}\sqrt[n]{n} - 1 \Rightarrow \lim_{n\to\infty}\sqrt[n]{n} \le 1 \qquad .... \quad (1)$$

But $n \ge 1$ $\Rightarrow \sqrt[n]{n} \ge \sqrt[n]{1} \Rightarrow \sqrt[n]{n} \ge 1 \Rightarrow \lim_{n\to\infty}\sqrt[n]{n} \ge 1$ .... (2)

Combine (1) and (2), $\lim_{n\to\infty}\sqrt[n]{n} = 1$.

(ii) $\sqrt[n]{\frac{n^3+n+1}{n^5+1}} > \sqrt[n]{\frac{n^3+n+1}{n^6}} > \sqrt[n]{\frac{n^3}{n^6}} = \left(\frac{1}{\sqrt[n]{n}}\right)^3$ and $\sqrt[n]{\frac{n^3+n+1}{n^5+1}} < \sqrt[n]{\frac{n^4}{n^5}} = \frac{1}{\sqrt[n]{n}}$

$\left(\frac{1}{\sqrt[n]{n}}\right)^3 < \sqrt[n]{\frac{n^3+n+1}{n^5+1}} < \frac{1}{\sqrt[n]{n}}$. By (i), $\lim_{n\to\infty}\left(\frac{1}{\sqrt[n]{n}}\right)^3 = 1$, $\lim_{n\to\infty}\frac{1}{\sqrt[n]{n}} = 1$

By Squeezing Principle, $\lim_{n\to\infty}\sqrt[n]{\frac{n^3+n+1}{n^5+1}} = 1$

(b) $f(x) = x^{1/x}$ $\Rightarrow$ $\ln f(x) = (1/x)\ln x$, $x > 0$ $\Rightarrow$ $\frac{1}{f(x)}f'(x) = -\frac{1}{x^2}\ln x + \frac{1}{x}\times\frac{1}{x} = \frac{1}{x^2}[1 - \ln x]$

$$\Rightarrow f'(x) = \frac{f(x)}{x^2}[\ln e - \ln x] \Rightarrow f'(x)\begin{cases} > 0 & , 0 < x < e \\ = 0 & , x = e \\ < 0 & , x > e \end{cases}$$

$\therefore$ $f(x)$ is increasing on $(0, e)$ $\Rightarrow$ $f(1) < f(2)$ $\Rightarrow$ $\sqrt[1]{1} < \sqrt[2]{2}$
$f(x)$ is decreasing on $(e, +\infty)$ $\Rightarrow$ $f(3) > f(4) > f(5) > \dots$ $\Rightarrow$ $\sqrt[3]{3} > \sqrt[4]{4} > \sqrt[5]{5} > ...$

Since $\sqrt[2]{2} \approx 1.414$, $\sqrt[3]{3} \approx 1.422$, therefore the greatest value among the sequence

$\{\sqrt[n]{n}\}$, n = 1, 2, … is $\sqrt[3]{3} \approx 1.422$

25. $y = \sin^{-1}x \quad \Rightarrow \quad \sin y = x \quad \Rightarrow \quad \cos y \dfrac{dy}{dx} = 1 \quad \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{\cos y} = \dfrac{1}{\sqrt{1-\sin^2 y}} = \dfrac{1}{\sqrt{1-x^2}}$

$y = \sin^{-1}x + (\sin^{-1}x)^2 \quad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-x^2}} + 2(\sin^{-1}x)\dfrac{1}{\sqrt{1-x^2}} = \dfrac{1}{\sqrt{1-x^2}}\left[1+2(\sin^{-1}x)\right]$ .... (1)

$$\frac{d^2y}{dx^2} = \frac{1}{\sqrt{1-x^2}}\left[\frac{2}{\sqrt{1-x^2}}\right] + \left[1+2(\sin^{-1}x)\right]\frac{x}{(1-x^2)^{3/2}} = \frac{1}{1-x^2}\left[2 + x\frac{1+2(\sin^{-1}x)}{\sqrt{1-x^2}}\right] = \frac{1}{1-x^2}\left[2 + x\frac{dy}{dx}\right]$$

$\therefore \quad (1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} = 2$ …. (2)

Differentiate (2) n-times and using Leibnitz's Theorem,

$$\left[(1-x^2)\frac{d^{n+2}y}{dx^{n+2}} + n(-2x)\frac{d^{n+1}y}{dx^{n+1}} + \frac{n(n-1)}{2}(-2)\frac{d^n y}{dx^n}\right] - \left[x\frac{d^{n+1}y}{dx^{n+1}} + n\frac{d^n y}{dx^n}\right] = 0$$

$\therefore \quad (1-x^2)\dfrac{d^{n+2}y}{dx^{n+2}} - x(2n+1)\dfrac{d^{n+1}y}{dx^{n+1}} - n^2\dfrac{d^n y}{dx^n} = 0$ …. (3)

Put n = 2r – 1 and x = 0 in (3),

$$(1-0^2)\frac{d^{2r+1}y}{dx^{2r+1}} - 0(2n+1)\frac{d^{2r}y}{dx^{2r}} - (2r-1)^2\frac{d^{2r-1}y}{dx^{2r-1}} = 0 \Rightarrow \frac{d^{2r+1}y}{dx^{2r+1}} = (2r-1)^2\frac{d^{2r-1}y}{dx^{2r-1}} \Rightarrow \frac{d^{2r+1}y}{dx^{2r+1}} = (2r-1)^2(2r-3)^2\frac{d^{2r-3}y}{dx^{2r-3}}$$

$$\Rightarrow \frac{d^{2r+1}y}{dx^{2r+1}} = (2r-1)^2(2r-3)^2\ldots3^2.1^2\frac{dy}{dx} = (2r-1)^2(2r-3)^2\ldots3^2.1^2\frac{1}{\sqrt{1-0^2}}\left[1+2(\sin^{-1}0)\right]$$

$$\Rightarrow \frac{d^{2r+1}y}{dx^{2r+1}} = (2r-1)^2(2r-3)^2\ldots3^2.1^2 = \left[\frac{(2r)(2r-1)(2r-2)(2r-3)\ldots3\cdot2\cdot1}{(2r)(2r-2)\ldots2}\right]^2 = \left[\frac{(2r)!}{2^r r!}\right]^2 = \frac{1}{2^{2r}}\left\{\frac{(2r)!}{r!}\right\}^2$$